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THE GROWTH OF A CIRCULAR HYDRORUPTURE CRACK IN AN ELASTIC SPACE WHEN A PLASTIC MATERIAL IS FORCED IN†

O. P. ALEKSEYENKO and A. M. VAISMAN

Novosibirsk

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An axisymmetric problem of the growth of a circular hydrorupture crack whose cavity is filled with a non-standard plastic flowing material is considered. The cases of complete and incomplete penetration are analysed. Prelimiting states of a crack of fixed dimensions are discussed.

IN HYDRORUPTURE problems, starting with the classical ones [1], the filling material is usually considered to be a fluid. But sometimes, especially in the case of experimental hydrorupture carried out to analyse the properties and state of a rock deposit, one is interested in fillers whose properties are close to ideal plasticity, for example some plastic greases [2]. Since there is practically no leakage and the resistance is independent of the flow velocity, it becomes possible to fix any state of the crack prior to measurement. Moreover, significant distortion of the edge contour can be eliminated,‡ which has been discovered in earlier experiments with plasticine [3].

Despite the obvious physical differences, the structure of the equations in the problem of a plastic layer in Bridgman anvils [4] is quite similar to that in the problem under consideration. However, the specific features of hydrorupture equations [5, 6] make it necessary to modify them due to the divergence of the iterative scheme.

1. Suppose that a circular planar crack of radius L at a given instant of time is developing in an elastic space because of a filling material with plastic properties being forced into the crack cavity from a point source at its centre. It is assumed that the flowing material does not lose contact with the crack edges and the pressure $P(X)$ vanishes on the boundary $X=\Gamma$ of the filled region (X is the radial coordinate of a cylindrical system of coordinates with origin at the crack centre).

For slow flow of an ideally plastic material in a narrow channel the pressure is distributed according to the equation [7]

$$dP/dX = -T_0/W \tag{1.1}$$

where W denotes the half-width of the crack at a distance X from the centre and T_0 is the friction stress on the crack edges. We note that the ideal plasticity condition may be violated, but if the rheology of the flowing material and its interaction with the cavity walls are such that for any radius greater than some small distance compared with L the shear stress in the filler does not

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exceed the order of magnitude of $T_0(X)$, then Eq. (1.1) in the boundary layer approximation follows directly from the general equations of equilibrium in terms of stresses. If (1.1) is satisfied, then for the system of equations stated below to be closed, it suffices to specify the law of sliding between the filler and the walls, in particular, in the form $T_0 = \text{const}$ used below, without any detailed properties of the flowing material inside the crack cavity.

The equation implies the estimate

$$P_*/\Gamma \approx T_0/W_* \tag{1.2}$$

for the characteristic values P_* and W_* of the pressure and width, respectively, In view of this estimate, in the case when the filler spreads into a region of the crack of radius comparable with L, the ratio of the shear and normal stresses has the order of magnitude of the characteristic strain W_*/L. This means that it suffices to consider only normal rupture cracks, i.e. neglect the shear stresses in the formulae expressing the displacements of crack edges in terms of the boundary forces applied to them. Otherwise, this would be equivalent to using quadratic corrections with respect to the deformations within the framework of the linear theory of elasticity.

The width of a normal rupture crack can be expressed in terms of the pressure on its edges by Sneddon's formula [8]. In problems concerned with hydrorupture it provides more convenient to use the elementary form

$$W(X) = -\frac{2}{\pi D}\int_0^\Gamma P'(X_1)\left\{(L^2 - X^2)^{1/2} - \int_{\max(X,X_1)}^{L} dX_2\left(\frac{X_2^2 - X_1^2}{X_2^2 - X^2}\right)^{1/2}\right\}dX_1 \tag{1.3}$$

Here $P'(X)$ is the radial pressure gradient, $\max(X, X_1)$ is the maximum of the two numbers in parentheses, and $D = \frac{1}{2}E(1-\nu)^2$ is a combination of the standard elastic constants.

As usual, the effect of the rock pressure P_g on the width of a crack lying in a horizontal plane is taken into account by subtracting

$$W_g = 2(\pi D)^{-1}(L^2 - X^2)^{1/2}P_g \tag{1.4}$$

from the right-hand side of (1.3) [8].

The relation between the asymptotic form of the width at the crack edge and the stress intensity factor K is given by [8]

$$\lim_{X\to L} W(X)(L-X)^{-1/2} = (2/\pi)^{1/2}K/D \tag{1.5}$$

There is a critical value $K = K_0$ assumed to be constant, corresponding to the crack growth. The integral law of conservation of mass for an incompressible flowing material

$$\Omega = 4\pi\int_0^\Gamma XWdX \tag{1.6}$$

closes the system of equations. Here Ω is the volume of the filling material forced into the crack.

2. It proves convenient to analyse the above system of equations in dimensionless form. For the coordinates X and Γ we change to linear units using the formulae $X = L\sin\varphi$ and $\Gamma = L\sin\gamma$. For the basic crack parameters we introduce scaling multipliers, denoted by the same letters with an asterisk

$$P_* = (\tfrac{1}{2}\pi D T_0)^{1/2},\quad W_* = LT_0/P_*,\quad \Omega_* = 4\pi L^2 W_*, \tag{2.1}$$

$$L_* = \pi(K_0/(2P_*))^2$$

We write the formulae of transition to dimensionless form as

$$P = P_* p, \quad P_g = P_* p_g, \quad L = L_* l, \quad \Omega = \Omega_* \omega \tag{2.2}$$

$$W = W_* \upsilon \cos\varphi, \quad W_g = W_* \upsilon_g \cos\varphi$$

The dimensionless variables are denoted by lower-case letters.

From (1.4), (2.1), and (2.2) we get

$$p_g = \upsilon_g = (\tfrac{1}{2}\pi D T_0)^{-1/2} P_g \tag{2.3}$$

We will now formulate the system of equations in terms of the new variables. Using (1.1), we eliminate P' from (1.3) and take into account the contribution W_g of the rock pressure P_g on the width of the crack. Changing to the dimensionless variables we obtain after some simple computations

$$\upsilon(\varphi) = -\upsilon_g + \int_0^\gamma \frac{A(\varphi_g, \psi)}{\upsilon(\psi)} d\psi, \quad \varphi \in [0, \pi/2], \quad \psi \in [0, \gamma] \tag{2.4}$$

$$A(\varphi, \psi) = 1 - \frac{1}{\cos\varphi} \int_{\max(\varphi,\psi)}^{\pi/2} \left(\frac{\sin^2\eta - \sin^2\psi}{\sin^2\eta - \sin^2\varphi} \right)^{1/2} \cos\eta \, d\eta \tag{2.5}$$

Note that the arguments φ and ϕ have different ranges of values, but if $\varphi > \gamma$, the corresponding part of $\upsilon(\varphi)$ can be easily determined from its values in the interval $[0, \gamma]$ using (2.4). In particular, for $\varphi = \pi/2$ we have

$$\upsilon_l = \upsilon(\pi/2) = -\upsilon_g + \int_0^\gamma \frac{1-\cos\psi}{\upsilon(\psi)} d\psi \tag{2.6}$$

In the interval $\varphi \in [0, \gamma]$ relation (2.4) is a non-linear integral equation in $\upsilon(\varphi)$, which must be solved. All the remaining parameters of the crack can be expressed in terms of υ. Indeed, changing to dimensionless variables with the aid of (2.1) and (2.2) in (1.1), (1.5), and (1.6), we obtain

$$p(\varphi) = \int_\varphi^\gamma \frac{d\psi}{\upsilon(\psi)} \quad 0 \leqslant \varphi \leqslant \gamma; \quad l = \upsilon_l^{-2}; \quad \omega = \int_0^\gamma \upsilon(\psi) \sin\psi \cos^2\psi \, d\psi \tag{2.7}$$

Since $A(\varphi, \psi) > 0$, we observe that γ decreases as the crack increases, the least admissible value γ_g of the degree of filling being attained in the limit as $l \to \infty$, when $\upsilon_l \to 0$. Then, by (2.6)

$$\upsilon_g = \int_0^{\gamma_g} \frac{1-\cos\psi}{\upsilon(\psi)} d\psi \tag{2.8}$$

and (2.4) takes the form

$$\upsilon(\varphi) = \int_0^{\gamma_g} \frac{A_g(\varphi, \psi)}{\upsilon(\psi)} d\psi, \quad A_g(\varphi, \psi) = A(\varphi, \psi) - (1 - \cos\psi) \tag{2.9}$$

3. We will now solve (2.4). As has been mentioned, the difficulty is due to the divergence of the standard iterative scheme

$$\upsilon_{n+1}(\varphi) = -\upsilon_g + \int_0^\gamma \frac{A(\varphi,\psi)}{\upsilon_n(\psi)} d\psi \quad (n = 0,1,2,\ldots)$$

As an illustration we present a numerical example for a weightless medium, because the approximations are bounded in this case. To fix our ideas, we set $\upsilon_g = 0$ and choose $\gamma = 1.5$ and $\upsilon_0 = 1$. After several initial iterations the odd and even approximations practically cease to change any more, but lie on different curves represented by the dashed curves in Fig. 1. The exact solution (the solid line) lies between them.

The following turns out to be an effective method of removing the divergence. We multiply both sides of (2.4) by $\upsilon(\varphi)$ and introduce the notation

$$A_0 = \upsilon(\varphi)\int_0^\gamma \frac{A(\varphi,\psi)}{\upsilon(\psi)} d\psi$$

The equation equivalent to (2.4) written in these terms is a quadratic algebraic equation. Its positive root is equal to

$$\upsilon = \tfrac{1}{2}(-\upsilon_g + \sqrt{\upsilon_g^2 + 4A_0}) \tag{3.1}$$

As has been demonstrated by computations, the successive approximation scheme for (3.1) converges. In the limit as $l \to \infty$ the equivalent form of (2.9) is

$$\upsilon = G_0^{1/2}, \quad G_0 = \upsilon(\varphi)\int_0^{\gamma_g} \frac{A_g(\varphi,\psi)}{\upsilon(\psi)} d\psi \tag{3.2}$$

In Figs 2–4 we present the computed dependence of the dimensionless pressure p_0, width υ_0 at the centre, the crack radius l, and the volume ω of the flowing material on the degree of filling γ. Curves 1–4 correspond to $\upsilon_g = 0$ (the case of a weightless medium), $\upsilon_g = 0.2463$, $\upsilon_g = 0.4424$, and $\upsilon_g = 1.023$. The dot–dash line marks the boundary of the domain of existence of the solution. As υ_g increases, the minimum admissible degree of filling γ is seen to increase, approaching $\pi/2$ in the limit. This means that for large external loads P_g the most typical situation is that of complete penetration of the crack by the filler ($\gamma = \pi/2$). As can be seen in Fig. 2, the pressure at the crack centre does not fall to zero when the crack increases without limit, as in the planar case [5], but tends to a finite limit; the larger the external load the higher the limit.

The pressure and crack width profiles, p and $\upsilon\cos\varphi$, for $\upsilon_g = 0.4424$ are presented in Fig. 5 for various values of γ. The dependence of the pressure p_0 at the centre on the crack radius l is shown in Fig. 6. Curves 1–4 correspond to the same values of υ_g as in Figs 2–4. The dot–dash line represents the curve $p_0 = p_0(\gamma, \upsilon_g)$ for $\gamma = \pi/2$. One can see that after the filler breaks away from the edge there is little change of pressure inside the crack as the crack increases; the larger the external load the smaller the change of pressure.

At any instant of time the state of the crack is determined by the pair of parameters (γ, υ_g). In practice it is more convenient to specify external parameters such as P_g and Ω. The transition from (γ, υ_g) to (P_g, Ω) is given by (2.3) and the expression

$$\omega l^3 = \Omega P_* / (4\pi L_*^3 T_0) \tag{3.3}$$

obtained using (2.1) and (2.2). The right-hand sides of (2.3) and (3.3) depend only on P_g, Ω and the constants of the problem, while the left-hand sides depend only on γ and υ_g. The corresponding values of υ, p and l can be found after determining γ and υ_g from (2.3) and (3.3). The dimensional parameters and variables can then be determined from (2.1) and (2.2).

4. The results can be generalized to the case of complete penetration of the crack by the filler and to the problem of a crack of fixed radius. We note that in the first case the pressure $P(L)$ on

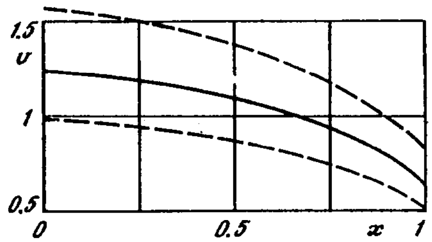

FIG. 1.

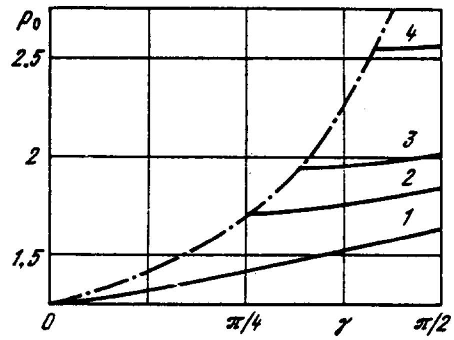

FIG. 2.

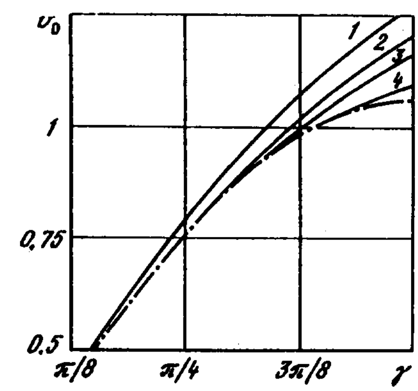

FIG. 3.

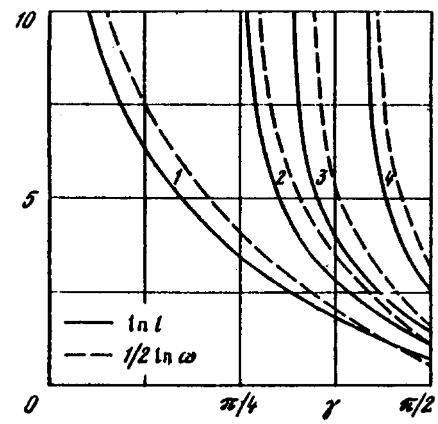

FIG. 4.

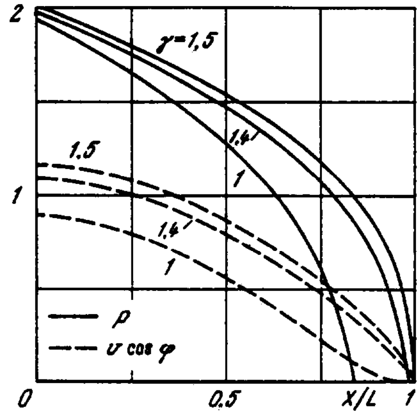

FIG. 5.

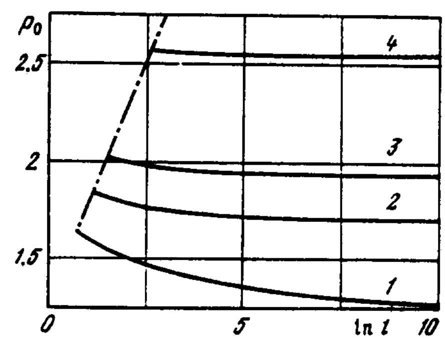

FIG. 6.

the crack edge corresponding to the given load P_g at infinity and the volume Ω of the filling material inside the gap is finite, in general. When solving the problem it proves more convenient to choose the formal governing parameter to be $P(L)$ rather than Ω. Strictly speaking, we mean by $P(L)$ the value obtained from the solution based on the approximation (1.1), rather than the actual value. In the problem in hand this approach would seem to require more precision, which could involve a significant change of pressure in a small neighbourhood near the edge, where (1.1) is not satisfied. But the pressure at the edge is of little interest in its own right. According to Sneddon's formula, the width of the crack is determined by the pressure curve as a whole. It can be shown that in relation to the crack radius the dimension of the domain considered near the edge is of the same order as the square of the characteristic strains in the elastic medium. Thus, when remaining within the framework of linear theory of elasticity, it makes no sense to increase the accuracy of the pressure variations at such short distances. Such corrections are therefore neglected in what follows.

The problem can be reduced to the previous case by the following method. We imagine that the state of strain of the elastic medium under consideration has been obtained in two stages. At first, the crack was filled under the reduced vertical load $P_g - P(L)$ at infinity. In this case, the load at the crack edges tends to zero as the edge is approached and one can apply the formulae of Section 2, setting $\gamma = \pi/2$. At the second stage a negative normal load $P(L)$ is added at infinity and on the crack edges, so that the deformations at the second stage are homogeneous and do not cause any displacements in the cut. This is essential because of the non-linearity of the integral equation determining the width. The resulting state of the medium and the crack obtained in this way will correspond to the specified values of P_g and $P(L)$.

We will change in the usual way to the dimensionless pressure on the edge, which we will denote by $p_l = p(\pi/2)$. It can be shown that for $p_l > 0$ the integral equation (2.4) can be transformed into

$$\upsilon(\varphi) = -(p_g - p_l) + \int_0^{\pi/2} \frac{A(\varphi,\psi)}{\upsilon(\psi)} d\psi$$

i.e. it can be reduced to the special case (2.4) when $\gamma = \pi/2$ and $\upsilon_g \to p_g - p_l$, and depends only on one parameter $p_g - p_l$. At the same time the total pressure distribution $p(\varphi)$ in the crack is determined, as it should be, by two parameters, for example, p_g and p_l. This distribution and, in particular, the value p_0 of the pressure at the centre can be computed from the formula

$$p(\varphi) = p_l + \int_\varphi^{\pi/2} \frac{d\psi}{\upsilon(\psi)}$$

which generalizes the first relation in (2.7).

Everything described above has been done under the assumption that $K_0 = \text{const}$. However, the basic dimensionless equation (2.4) does not contain any information on the crack dimensions L or the value of the stress intensity factor K. This enables us to use the equation also in the case when, for example, K_0 depends on the velocity of motion of the edge [9]. Indeed, the problem can be solved for any instant of time and any value K_0. In particular, the dimensions of the crack and, consequently, the velocity of the edge can be determined at any instant of time. The problem of matching K_0 with the velocity of the edge can be reduced to a purely algebraic problem.

Equation (2.4) is also applicable in the case when the crack is in a subcritical state with fixed radius. The only difference is that K rather than L will be the variable quantity when one changes to dimensional variables in the corresponding formulae. One can introduce K_* and k in place of L_* and l using the formulae

$$K = K_* k, \quad K_* = \sqrt{2LDT_0}$$

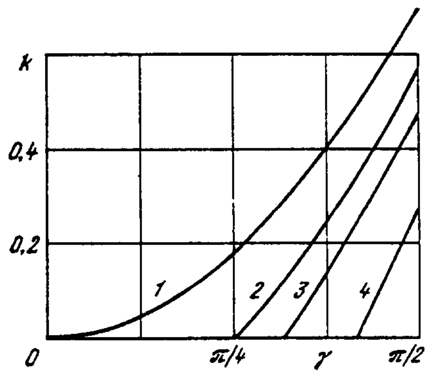

FIG. 7.

Using (1.5) and the formulae for changing to dimensionless variables, we find that $k = \upsilon_l$. In Fig. 7 we show the dependence of the parameter k, characterizing the stress concentration on the crack edge, on the degree of filling γ. Curves 1–4 correspond to the same values υ_g as in Figs 2–6. The intersection of each curve with the abscissa axis indicates that the crack fails to be fully open for smaller values of γ, i.e. the edges close smoothly at a radius smaller than the dimensions of the crack.

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